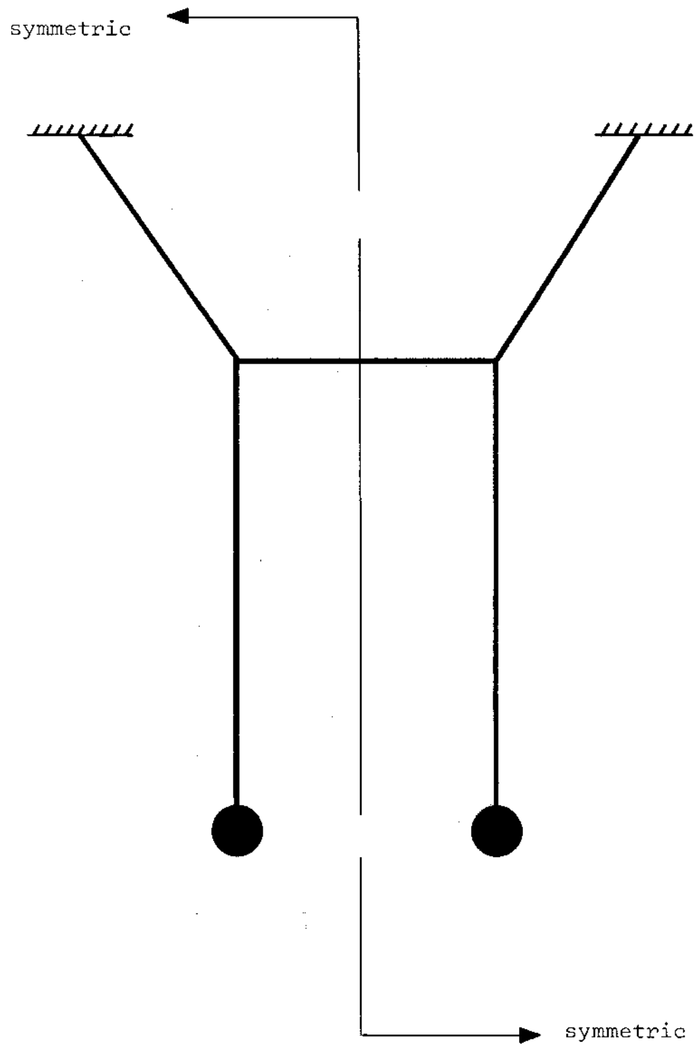


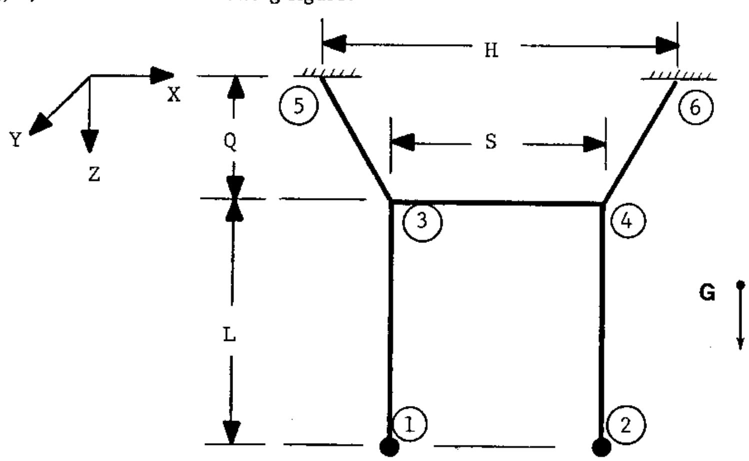
THE COMPOUND PENDULUM

An Essay on the Compound Pendulum

The problem of predicting the motion of the compound pendulum, shown opposite, was presented to my freshman college physics class as unsolved; and so it remained, at least in my thinking, for the next ten years. This is the season when a youth can imbibe such elegant syntheses of man's understanding as the following: The image of pulses of light as orthogonally alternating electric and magnetic fields, which measure the universe by the speed of their advance; the eccentrically periodic ascendance and decline of pagan and puritan instincts that we record as human history; and the endless precession of DNA's twin coils, carrying the history of life's evolution on our planet. Each of these phenomena is a manifestation of what we would call the working-out of a dualistic principle with time.

The fasicination with the compound pendulum seems to owe to its being the simplest imaginable generator of this sort of movement. Even while watching the pendulum's motion, it is hard to believe that such a simple device could possibly behave in ways that are analogous to the several complex and important processes of which it may be thought symbolic. Perhaps it is a respect for the imprint that the image of dualism-in-motion has made on recent thought that has led to the establishing of mechanics of the compound pendulum.

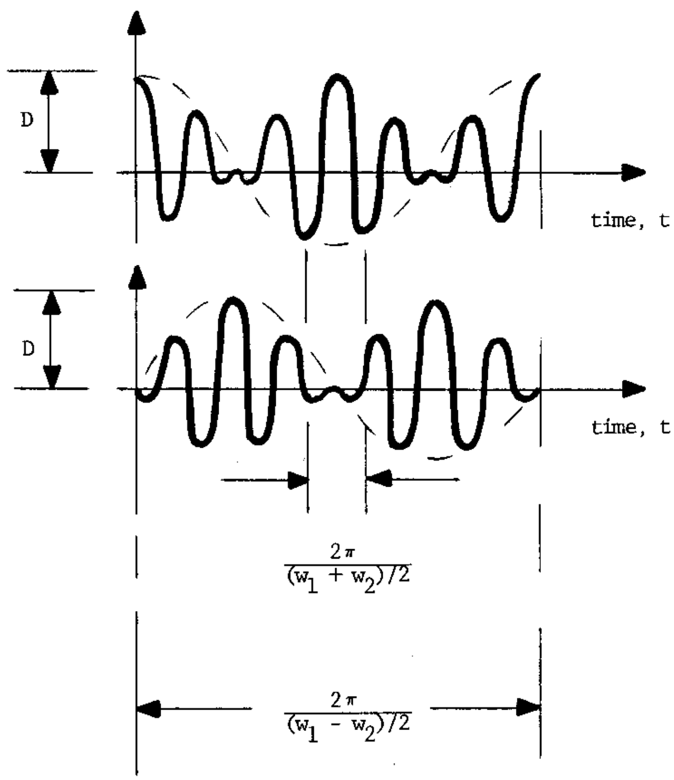
Article 1: Notation, co-ordinate frames, and sign conventions. When at rest, a symmetric compound pendulum can be fully described by as few as four parameters; viz the Q, L, S and H of the following figure:



The plane in which these dimensions lie is called the "X-Z" plane: The "X" direction is given by the line joining the two support locations of the pendulum; and the "Z" direction is aligned with gravity, G. The "Y" direction is perpendicular to this plane, and the positive sense of each direction is chosen so as to give us a right-handed frame of reference.

The circled integers (1) through (6) refer to this system's node points. In our analysis, we will refer to the variables of position, P, acceleration, A, and reaction, R, which are developed at these nodes and which act in the three coordinate directions. The variable M will denote the mass of either pendulum weight.

Article 2: The equation of the bi-harmonic oscillator. The characteristic motion of the compound pendulum is that of bi-harmonic oscillation, such as diagrammed below.



This is the form of motion that will be observed when the movement of the pendulum is initiated by displacing one mass along the X- or Y- axis. A general algebraic description of the two wave forms shown above would be:

$$\text{Eqa. 2-1:} \quad P_1 = D\cdot\cos\left[\left(\frac{w_1 + w_2}{2}\right)\cdot t\right]\cdot\cos\left[\left(\frac{w_1 - w_2}{2}\right)\cdot t\right]$$

$$\text{Eqa. 2-2:} \quad P_2 = D\cdot\sin\left[\left(\frac{w_1 + w_2}{2}\right)\cdot t\right]\cdot\sin\left[\left(\frac{w_1 - w_2}{2}\right)\cdot t\right]$$

where D corresponds to the value of P_1 when time, t, is equal to zero; where P_2 is zero at t=0; and where w_1 and w_2 define the periods of the bi-harmonic cycles and epi-cycles, as shown in the preceding figure. The essence of solving the pendulum problem can be said to lie in establishing the internal frequencies, w_1 and w_2 on the basis of the dimensions of a particular compound pendulum. Let us proceed to isolate these natural frequencies empirically before presenting a rigorous solution, which requires more algebra than would be justifiable on the basis of an ordinary level of curiosity.

If, in later analysis, we are able to establish that the mutual effects of the movements of the two halves of the pendulum can be superimposed upon one another by simple addition, then we can elaborate Equations 2-1 and 2-2 to the point of specifying the motions which proceed from all initiating displacements, D_1 and D_2, of the two masses. Retaining, for the momemt, our reference to motions which occur entirely in the X- or in the Y- direction, the general equations for the movements in either coordinate direction would be:

$$\text{Eqa. 2-3:} \quad P_1 = D_1\cdot\cos\left[\left(\frac{w_1 + w_2}{2}\right)\cdot t\right]\cdot\cos\left[\left(\frac{w_1 - w_2}{2}\right)\cdot t\right] + D_2\cdot\sin\left[\left(\frac{w_1 + w_2}{2}\right)\cdot t\right]\cdot\sin\left[\left(\frac{w_1 - w_2}{2}\right)\cdot t\right]$$

$$\text{Eqa. 2-4:} \quad P_2 = D_2\cdot\cos\left[\left(\frac{w_1 + w_2}{2}\right)\cdot t\right]\cdot\cos\left[\left(\frac{w_1 - w_2}{2}\right)\cdot t\right] + D_1\cdot\sin\left[\left(\frac{w_1 + w_2}{2}\right)\cdot t\right]\cdot\sin\left[\left(\frac{w_1 - w_2}{2}\right)\cdot t\right]$$

Earlier we observed that the bi-harmonic behavior mode is initiated by giving one or the other mass an initial displacement and allowing the other mass to begin its motion from rest. Let us perform a few imaginary experiments in which we contrive the motion-inducing displacements, D_1 and D_2, so as to produce simple harmonic behaviors wherein the separate identities of the natural frequencies, w_1 and w_2 will become manifest. Intuition would have it that setting $D_1 = D_2$ or $D_1 = -D_2$ will produce some interesting results. Let us see.

When $D_1 = D_2$ (=D) obtains, Equations 2-3 and 2-4 become:

$$\text{Eqa. 2-5:}\quad P_1 = P_2 = D\bullet\left\{\cos\left[\left(\frac{w_1 + w_2}{2}\right)\cdot t\right]\cdot\cos\left[\left(\frac{w_1 - w_2}{2}\right)\cdot t\right] + \sin\left[\left(\frac{w_1 + w_2}{2}\right)\cdot t\right]\cdot\sin\left[\left(\frac{w_1 - w_2}{2}\right)\cdot t\right]\right\}$$

Now we apply the formulae for the sums and differences of angles to the expression in the brackets of this equation. Recalling that

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi; \qquad \cos(-\theta) = \cos\theta$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi; \qquad \sin(-\phi) = -\sin\phi$$

we can transform Equation 2-5 into:

$$\begin{aligned}\text{Eqa. 2-6:}\quad P_1 = P_2 = D\bullet\Big\{&\left[\cos\frac{w_1\cdot t}{2}\cdot\cos\frac{w_2\cdot t}{2} - \sin\frac{w_1\cdot t}{2}\cdot\sin\frac{w_2\cdot t}{2}\right]\\ \bullet&\left[\cos\frac{w_1\cdot t}{2}\cdot\cos\frac{w_2\cdot t}{2} + \sin\frac{w_1\cdot t}{2}\cdot\sin\frac{w_2\cdot t}{2}\right]\\ +&\left[\sin\frac{w_1\cdot t}{2}\cdot\cos\frac{w_2\cdot t}{2} + \cos\frac{w_1\cdot t}{2}\cdot\sin\frac{w_2\cdot t}{2}\right]\\ \bullet&\left[\sin\frac{w_1\cdot t}{2}\cdot\cos\frac{w_2\cdot t}{2} - \cos\frac{w_1\cdot t}{2}\cdot\sin\frac{w_2\cdot t}{2}\right]\Big\}\end{aligned}$$

Multiplying as indicated brings us to:

$$\begin{aligned}\text{Eqa. 2-7:}\quad P_1 = P_2 = D\bullet\Big[&\cos^2\left(\frac{w_1\cdot t}{2}\right)\cdot\cos^2\left(\frac{w_2\cdot t}{2}\right) - \sin^2\left(\frac{w_1\cdot t}{2}\right)\cdot\sin^2\left(\frac{w_2\cdot t}{2}\right)\\ +&\sin^2\left(\frac{w_1\cdot t}{2}\right)\cdot\cos^2\left(\frac{w_2\cdot t}{2}\right) - \cos^2\left(\frac{w_1\cdot t}{2}\right)\cdot\sin^2\left(\frac{w_2\cdot t}{2}\right)\Big]\end{aligned}$$

Factoring with an eye to $\cos^2(w_1\cdot t/2) + \sin^2(w_1\cdot t/2) = 1$, will yield

$$\text{Eqa. 2-8:}\quad P_1 = D\bullet\left[\cos^2(w_2\cdot t/2) - \sin^2(w_2\cdot t/2)\right] = P_2$$

Observing that $\cos^2\theta - \sin^2\theta = \cos 2\theta$ we finally arrive at:

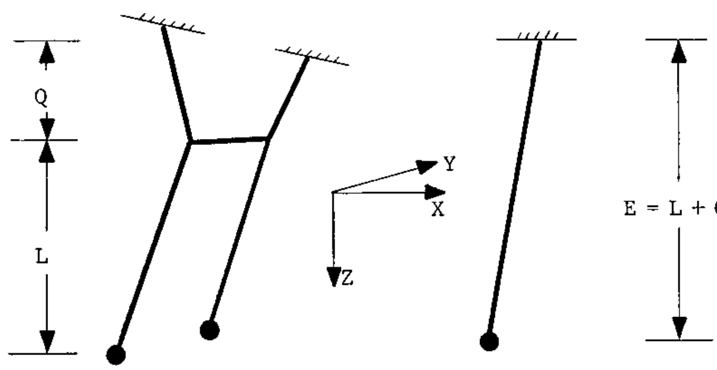
Eqa. 2-9: $P_1 = P_2 = D \cdot \cos(w_2 \cdot t)$

which tells us that the smaller of the two internal frequencies of the X- or the Y- motion of a compound pendulum, w_2, governs the simple harmonic motion which proceeds from the equal initiating displacement of both masses, $D_1 = D_2$. An equally tedious process, employing the same trigonometric identities in the same way, will show the equal and opposite initial displacements of the pendulum's two masses, $D_1 = - D_2$, will yield a similar equation in the larger of the internal frequencies, w_1:

Eqa. 2-10: $P_1 = - P_2 = D \cdot \cos(w_1 \cdot t)$

Now let us bring Equations 2-9 and 2-10 to bear on the movements which observe along the X- and Y- axes when either of these sets of initial conditions obtain.

Article 3: Observations of motion along the Y-axis. Taking the case of equal initial displacements along the Y-axis first, we observe that the two masses behave identically to the single mass of a simple pendulum with a total string length of Q + L:



The equation for the motion of a simple pendulum, such as on the right above,

Eqa. 3-1: $P = D \cdot \cos(w \cdot t)$

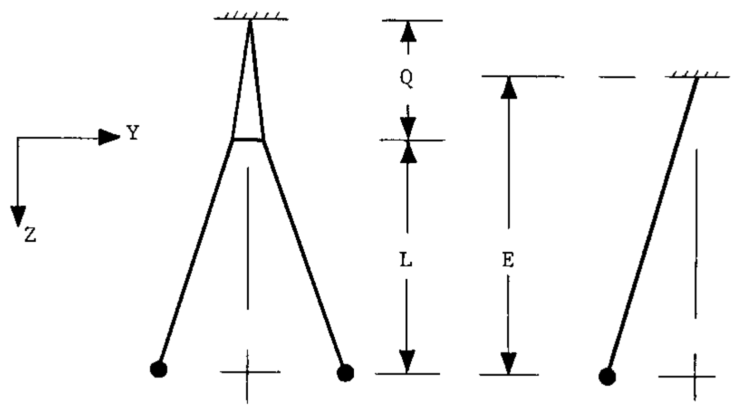
will be derived later as a part of our formal statement of the mechanics of the compound pendulum. At that time, it will be shown that the frequency, w, of a simple pendulum is given by the square root of the intensity of the local gravity field, G, divided by the length, E, of the string which joins the pendulum weight to its support:

Eqa. 3-2: $w = \sqrt{G/E}$

A comparison of Equations 2-9 and 3-1 together with our anology to the simple pendulum, above, will let us transform Equation 3-2 into the equation for the smaller of the two frequencies which govern motion along the Y-axis of a compound pendulum:

Eqa. 3-3: $w_{2y} = \sqrt{\frac{G}{L + Q}}$

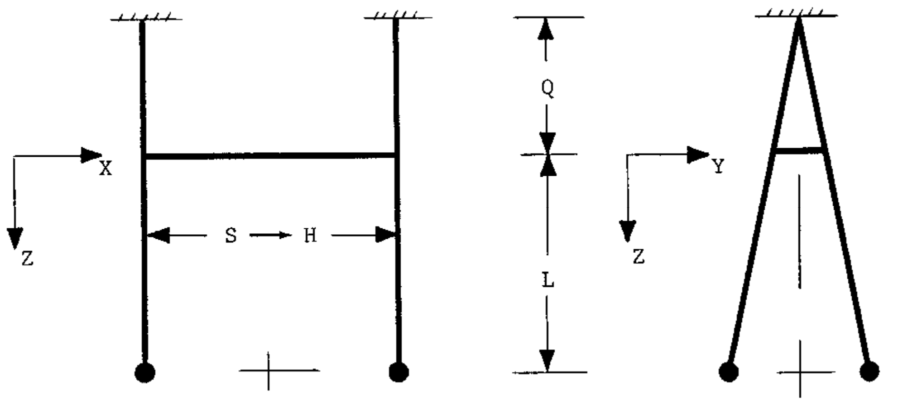
The analogy to a simple pendulum of an appropriate "equivalent length" is also helpful in determining the larger of the two frequencies that govern motion along the Y-axis, w_{1y}. Recall from Equation 2-10 that this is the frequency of the simple harmonic motion which follows from equal initializing displacements:



In the figure above, the pendulum weights move in a manner which always keeps them 180° out of phase. The period of this motion is the same as that of a simple pendulum with an equivalent length, E, that is somewhere between L and L + Q:

Eqa. 3-4: $L \leq E \leq L + Q$

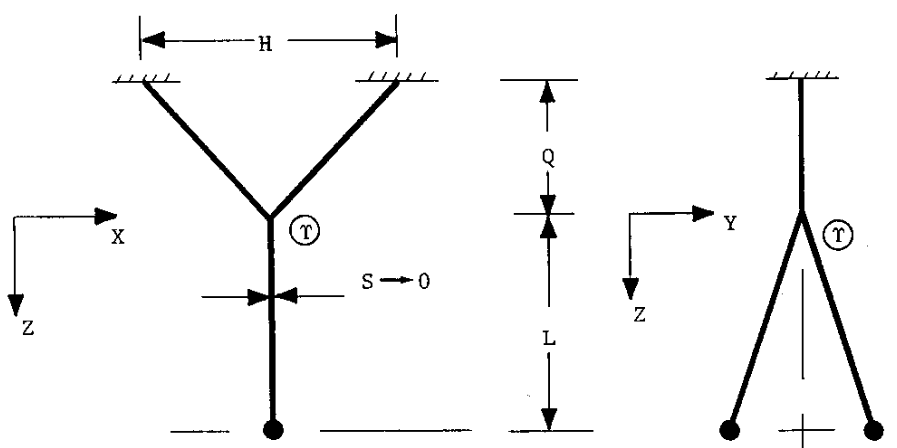
In order to determine E for the initial conditions implied above, we will have to examine the extreme case of the value of S, the length of the horizontal string which joins the two halves of the compound pendulum. As S ⟶ H, the two halves of the pendulum become increasing independent, and the inner and outer cycles merge to a single harmonic wave form:



Here the simple pendulum which produces the analogous motion has an equivalent length equal to L + Q:

Eqa. 3-5: $E = L + Q$ for $S \longrightarrow H$

Let us now repeat our experiment with a symmetric compound pendulum that has been set in motion by equal and opposite dispalcements along the Y-axis, but with S approaching zero:



Here we observe the inner and outer cycles again converging to a simple harmonic wave form, while the equally opposed forces that are developed through the symmetric movements of the two masses tend to peg node (T) to a fixed point at the distance Q from the supports. In this case, an anologous simple pendulum would have an equivalent length of L:

Eqa. 3-6: $E = L \text{ for } S \longrightarrow 0$

We may also observe that this equivalent length will also obtain as Q approaches zero.

At this point, we have nothing to lose by adding the assumption of linearity in the realtionship between E and S to our growing list of what must be proven later. The postulate of a linear linkage between Equations 3-5 and 3-6 suggests the following equation for the equivalent length, E, of the simple pendulum which moves in a manner that is analogous to the movements of a compound pendulum that has been initialized with equal and opposite displacements along the Y-axis:

Eqa. 3-7: $E = L + S \cdot Q/H$

So we now have an empirically derived opinion for the nature of the greater of the two frequencies which govern motion in the Y-direction:

Eqa. 3-8: $w_{1y} = \sqrt{\frac{G}{L + S \cdot Q/H}}$

Having established w_{1y} and w_{2y} (in Equation 3-3) on the basis of experiments, let us now propose an empirical method for testing these findings. Laboratory procedures for recording the instantaneous positions, velocities, and accelerations of moving bodies are far from trivial, and require much more apparatus than would be in keeping with the spirit of using such a straight-forward instrument as a simple pendulum to infer the behaviors of the compound pendulum. Let us return to the observation of bi-harmonic motion which introduced Article 2 in order to contrive another experiment that can easily prove or disprove our work to this point. Please recall that this motion follows from an initial displacement of only one of the pendulum weights while the other is left to begin movement from its rest position. We have postulated the following equation for the movement of one of the node points denoting the location of a pendulum weight:

Eqa. 2-1: $P_1 = D \cdot \cos\left[\left(\frac{w_1 + w_2}{2}\right) \cdot t\right] \cdot \cos\left[\left(\frac{w_1 - w_2}{2}\right) \cdot t\right]$

And we have observed that frequencies, w_1 and w_2, in this equation correspond to the periods of the inner and outer cycles of the bi-harmonic motion described at the outset of Article 2.

So it can be said that the frequencies, w_1 and w_2, of Equation 2-1 are apparent in an easily made observation such as the number of times one of the pendulum weights will oscillate in between those occasional cycles when it appears to be completely at rest. A count, N, of number of pulses per half cycle is given by ratio of half the larger period to the shorter period:

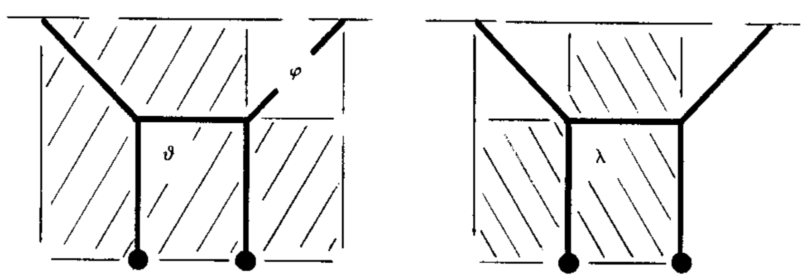
Eqa. 3-9: $$N = \frac{1}{2} \cdot \frac{\frac{2\pi}{(w_1 - w_2)/2}}{\frac{2\pi}{(w_1 + w_2)/2}} = \frac{1}{2} \cdot \frac{w_1 + w_2}{w_1 - w_2}$$

In order to evaluate this equation, let us re-state our Equations 3-3 and 3-8 in terms of a common demoninator:

Eqa. 3-10: $$w_{1y} = \sqrt{\frac{G(\vartheta + \varphi)}{L\vartheta + Q\lambda}}$$

Eqa. 3-11: $$w_{2y} = \sqrt{\frac{G(\vartheta - \varphi)}{L\vartheta + Q\lambda}}$$

where the Greek letters have the physical significance of coresponding to various areas in the plane of the compound pendulum:



Substituting Equations 3-10 and 3-11 into Equation 3-9 will quickly yield:

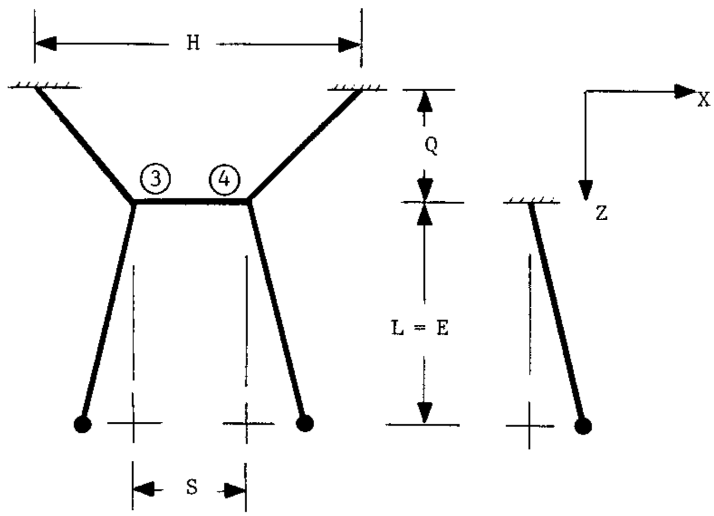
Eqa. 3-12: $$N = \frac{\vartheta + \sqrt{\vartheta^2 + \varphi^2}}{2\varphi}$$

which reduces to:

Eqa. 3-13: $N \approx \vartheta/\varphi$

for cases where φ is much less than ϑ, which is typically the case. So, we offer Equation 3-13 as a statement of our findings which can readily disprove our work so far—that is unless we have been fortunate enough to have been correct.

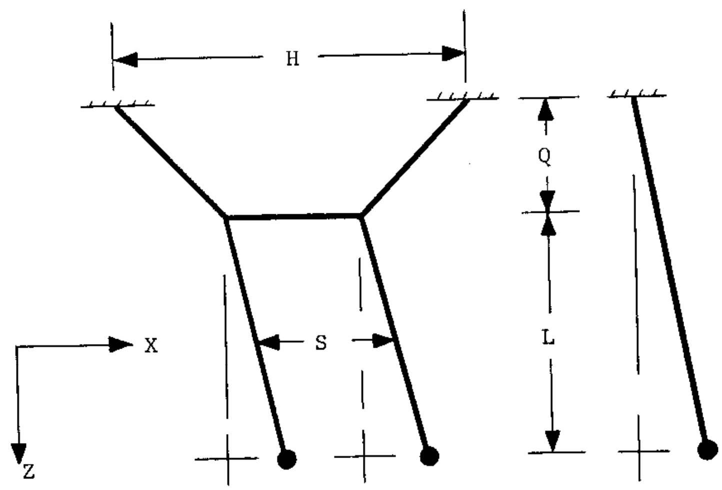
Article 4: Observations of motion along the X-axis. Our work in the two previous articles enables us to go directly to experiments which will reveal the natural frequencies of motion along the X-axis of the compound pendulum. We know that the larger of these two frequencies, w_{1x} will correspond to the simple harmonic motions of both halves of a compound pendulum that is set in motion by equal and opposite initial displacements of the two pendulum weights along the X-axis:



Here again we observe that the equally opposed forces developed in the symmetric movements of the pendulum's masses will peg nodes (3) and (4) motionless in their rest positions, and that the only dimension which enters into the dynamics of this system is L, hence:

$$\text{Eqa. 4-1:} \quad w_{1x} = \sqrt{\frac{G}{L}}$$

We would expect w_{2x} to reveal itself when the initial X-displacements are alike in both magnitude and direction:



Here we observe displacements of the entire plane of the compound pendulum at a frequency which corresponds to that of a simple pendulum with the same total distance between its point of support and the pendulum weight. The frequency of this motion is given by:

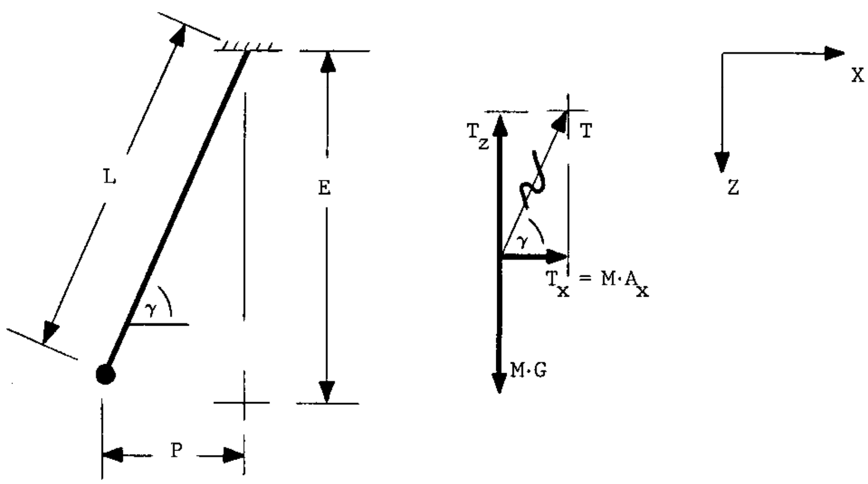
$$\text{Eqa. 4-2:} \quad w_{2x} = \sqrt{\frac{G}{L + Q}}$$

These results can be tested by the means of predicting the number of pulses per half cycle that was developed at the end of the last article. The corresponding equation,

$$\text{Eqa. 4-3:} \quad N = \frac{2L + Q + 2\sqrt{L^2 + LQ}}{2Q}$$

has no apparent elegance, especially in relation to the corresponding equation for Y-motion. Note also that the horizontal dimensions, H and S, of the compound pendulum do not enter into the dynamics of motions that take place entirely along the X-axis, this gives us some important cirtiera for our search into the principles that we require for a formal development of the parameters and equations which govern the motion of the compound pendulum. Since Y-motion does not distort the figure which the pendulum presents to the X-Z plane, it cannot cause the horizontal components of reaction to vary from their rest values; X-motion, on the other hand, can only be governed by the impulses which arise from the differences between these reactions. Hence our final observation would be to effect that the dimensions H and S are significant to the compound pendulum only in that they specify the rest values of the X-components of the support reactions: These dimensions enter into Y-motion because the rest value of X-reactions must somehow enter into the values which are taken on by the Y-reactions during movements along the Y-axis; the H and S dimension do not enter into purely X-motion because the rest values of the X-reactions are equal and opposite and will, therefore, cancel out when the net impulse from these reactions is determined.

Article 5: Principles of Analysis. Our formal derivation of the equations and parameters which govern the movements of the compound pendulum will be based on extensions of the same principles that are necessary to derive the equation of motion for a simple pendulum. An elementary free body diagram is sufficient to convey a sense of the flow of forces in a simple pendulum:



In this figure, a pendulum of length L supports a weight of mass M in a gravitational of intensity G. Displacements from rest along the X-axis are measured by the variable P; accelerations in this direction are denoted by the variable A, which is the second derivative of P with respect to time; and the components of the tension in the string, T, resolve into T_z and T_x along the coordinate axes. Our basic principle for analyzing this figure is simply the definition of a string as it is used in structural analysis, viz: An idealized "string" offers no resistance to forces that would tend to bend it. Since we observe that the string which joins the pendulum weight to its support remains straight during oscillations, we must assume that all the forces on the pendulum weight resolve into the axis of the string. The foregoing can be summarized in saying that the angle, γ , at which the pendulum is inclined must coincide with the inclininations of the tension vector, T, at all times. Mathematically, this condition can be expressed as:

Eqa. 5-1: $$\frac{T_x}{T_z} = \frac{P}{E}$$

The force diagram allows us to read off the following equations for the components of tension:

Eqa. 5-2: $T_x = M \cdot A$

Eqa. 5-3: $T_z = - M \cdot G$

Substituting these components of tension into Equation 5-1 and re-arranging will yield:

Eqa. 5-4: $P = - \frac{E}{G} \cdot A$

Since A is defined as the second time derivative of P, this would be a fairly tractable differential equation until be observe that E is related to P by way of Pythagoras:

Eqa. 5-5: $E^2 = L^2 - P^2$

At this point, the typical procedure is to note that $\sqrt{L^2 - P^2}$ is not greatly different from L when P is small in comparison to L. With this provision, we can substitute L for E in Equation 5-4 to arrive at the following equation of motion for the simple pendulum:

Eqa. 5-6: $P = -A \cdot G/L$

The equation above defines the simple harmonic motion which is generally thought of as being epitomized in the simple pendulum. (It may be intersting to note that actual pendula are in strict conformance with this equation only to the extent that they exhibit no motion at all.) The behavior modes to which the varible P is constrained by Equation 5-6 can be shown, by direct substitution, to be fully expressed by:

Eqa. 5-7: $P(t) = D \cdot \cos(w \cdot t + \delta)$

where D arises from a displacement of the pendulum weight along the X-axis and w is given by $\sqrt{G/L}$. The parameter δ allows for motions that are initiated by imparting an initial velocity to the pendulum weight. Having mentioned the possibility of motions that are originated in this manner, we are going to drop the parameter δ from consideration at this point. Our feeling is that the motions of the compound pendulum can be made fully apparent through initiating stimuli that are entirely confined to displacements of the two pendulum weights at time = 0; and that an exhaustive consideration of all the possible initial conditions would quadruple the number of equations and parameters that would have to be treated while adding nothing to the entertainment value of our presentation. Thus, our references to the equation for the motion of a simple pendulum will be as follows:

Eqa. 5-8: $P(t) = D \cdot \cos(w \cdot t)$

A passing observation may be made to the effect that our treatment of the simple pendulum allows no deviations in the vertical component of the support reaction from its rest value of M·G: Since our analyses have lost sight of vertical displacements of the pendulum weight when E was equated to L, there are no vertical accelerations to justify any unbalanced forces along this axis. Ostensibly, the deviations of the vertical support reaction would be relatively small, just as P is small relative to L, so that implications of our observation have little significance for the simple pendulum. But, when we transfer our understanding of the simple pendulum to the analysis of the compound pendulum, the simplifications which follow from the assumption of zero vertical displacements become significant in a number of ways.

First, it should be noted that, if nodes (1) and (2) are not displaced vertically in either X- or Y-motion, then nodes (3) and (4) will not be displaced vertically either. This means that the distance between nodes (3) and (4) does not change, and that the X-components for the positions of these nodes must, therefore, be identical at all times. A second manifestation of the assumption of zero vertical displacements is that all motions in the X-Y plane can be decomposed along the X- and Y-axes in a manner that shows the movements along one axis to be independent of the movements taking place along the other axis. On first consideration, this notion might appear to be arguable from the standpoint of a balance of movements in the X-Y plane, where it is clear that the accelerations and displacements along the two axes do interact to specify the Y-components of reaction in ways which differ according to differing degrees of the displacements.

There are a number of different ways to approach a point of view in which these interactions appear to be of the "second-order" variety that were dismissed in the analysis of the simple pendulum: Our initial observation might be to the effect that all variations in the moment arms owing to displacements in the X-Y plane are small in comparison to the dimensions H and S, and that the forces developed in this plane are small in comparison the the constant pull of gravity on the pendulum weights. (Please consult the diagram of forces and displacements folded into the opposite page.) This approach is essentially a translation of the simplifying assumptions that were made for a simple pendulum to a horizontal plane. Another appraoch would originate in the observation that the assumption of zero vertical displacements confers the identity of a linear, elastic system on the compound pendulum: In displacing a pendulum weight in the horizontal plane, one encounters a resisting force that is proportional to the amount of the displacement (see Equation 5-4). Hence, we should, again, expect the principle of structural superposition to apply; which it does in that all changes in the length of moment arms in the X-Y plane will be exactly off-set by an opposing variation in the force which acts through that moment arm.

The third manifestation of the assumption of zero vertical displacements for a compound pendulum is that the vertical components of reaction cannot change as a result of the motions which take place in the X-Y plane. This notion should be problematic only for X-motion, since movements along the Y-axis can be discounted in their effect on vertical reaction components on the basis of the same reasoning that was presented for a simple pendulum. But it is far from clear that the displacements which take place in the X-Z plane have been constrained so that they will always conspire with the corresponding array of forces in a manner which keeps the burden of supporting the pendulum weights equally distributed between the two reaction points. Our concern here is that the formal development of the equations of motion for the compound pendulum will consider the X- and Y-components of motion separately, and then assert that the principle of structural superposition allows us to consider that these equations hold regardless of what may be happening along a mutually orthogonal axis. Our preparation for such an assertion would be entirely complete at this point, except that our development of the equations for Y-motion depends on constant vertical reactions. Thus, the discerning reader may not allow that the equations for Y-motion hold when X-motion is present.

In order to avoid the possibility of appearing to use the principle of structural superposition in a facile or merely circular manner, let us accept the obligation to demonstrate the constant nature of the vertical reactions as part of the development of our equations for X-motion.

Article 6: The equations of motion. There are essentially three different considerations which govern the forms of movement that the compound pendulum can demonstrate. The first of these are "pendulum" equations that are analogous that Equation 5-6. These equations require a reference to the positions of nodes (3) and (4) in order to relate the accelerations of nodes (1) and (2) to the inclinations to the strings which join the respective nodes:

Eqa. 6-1: $A_{1x} = -G \cdot (P_{1x} - P_{3x})/L$

Eqa. 6-2: $A_{2x} = -G \cdot (P_{2x} - P_{4x})/L$

Eqa. 6-3: $A_{1y} = -G \cdot (P_{1y} - P_{3y})/L$

Eqa. 6-4: $A_{2y} = -G \cdot (P_{2y} - P_{4y})/L$

These equations are an expression of the structural definitions of a string as being incapable of resisting forces which tend to bend it. Four equations are required to translate this notion from the simple pendulum of Article 5 to the compound pendulum: One equation for each of the two extreme nodes in each of the two coordinate directions that define the plane of motion.

The structural definition of a string imposes a similar set of constraints on the relationships between the support reactions and the inclinations of the strings at the points of support. A mathematical expression of the notion that the forces acting at nodes (5) and (6) must be resolved to the axis of the string which is connected to that node would be as follows:

Eqa. 6-5: $R_{5x}/R_{5z} = [(H - S)/2 + P_{3x}]/Q$

Eqa. 6-6: $R_{6x}/R_{6z} = [(H - S)/2 - P_{4x}]/Q$

Eqa. 6-7: $R_{5y}/R_{5z} = P_{3y}/Q$

Eqa. 6-8: $R_{6y}/R_{6z} = P_{4y}/Q$

A third group of equations arises from the definition of structural support, which can be said to generate just the critical amount of force that is necessary to keep the support point stationary at all times. These equations state the overall balance of the forces acting along each coordinate axis, and the requirements that the resultant of all moments in each plane of motion be zero. The equation which assures that nodes (5) and (6) do not rotate in the Y-Z plane is

$$\text{Eqa. 6-9:} \quad \sum M_{⑤} {+}\circlearrowright = 0 = -M\cdot(A_{1y} + A_{2y})\cdot(L + Q) - M\cdot G\cdot(P_{1y} + P_{2y})$$

The same consideration is the X-Z plane yields:

$$\text{Eqa. 6-10:} \quad \sum M_{⑤} {+}\circlearrowright = 0 = R_{6z}\cdot H - M\cdot(A_{1x} + A_{2x})\cdot(L + Q) - M\cdot G\cdot\left[\frac{H + S}{2} + P_{2x} + \frac{H - S}{2} + P_{1x}\right]$$

In order for the supports to remain stationary in relation to the Y-axis, the reaction components must obey the following equation:

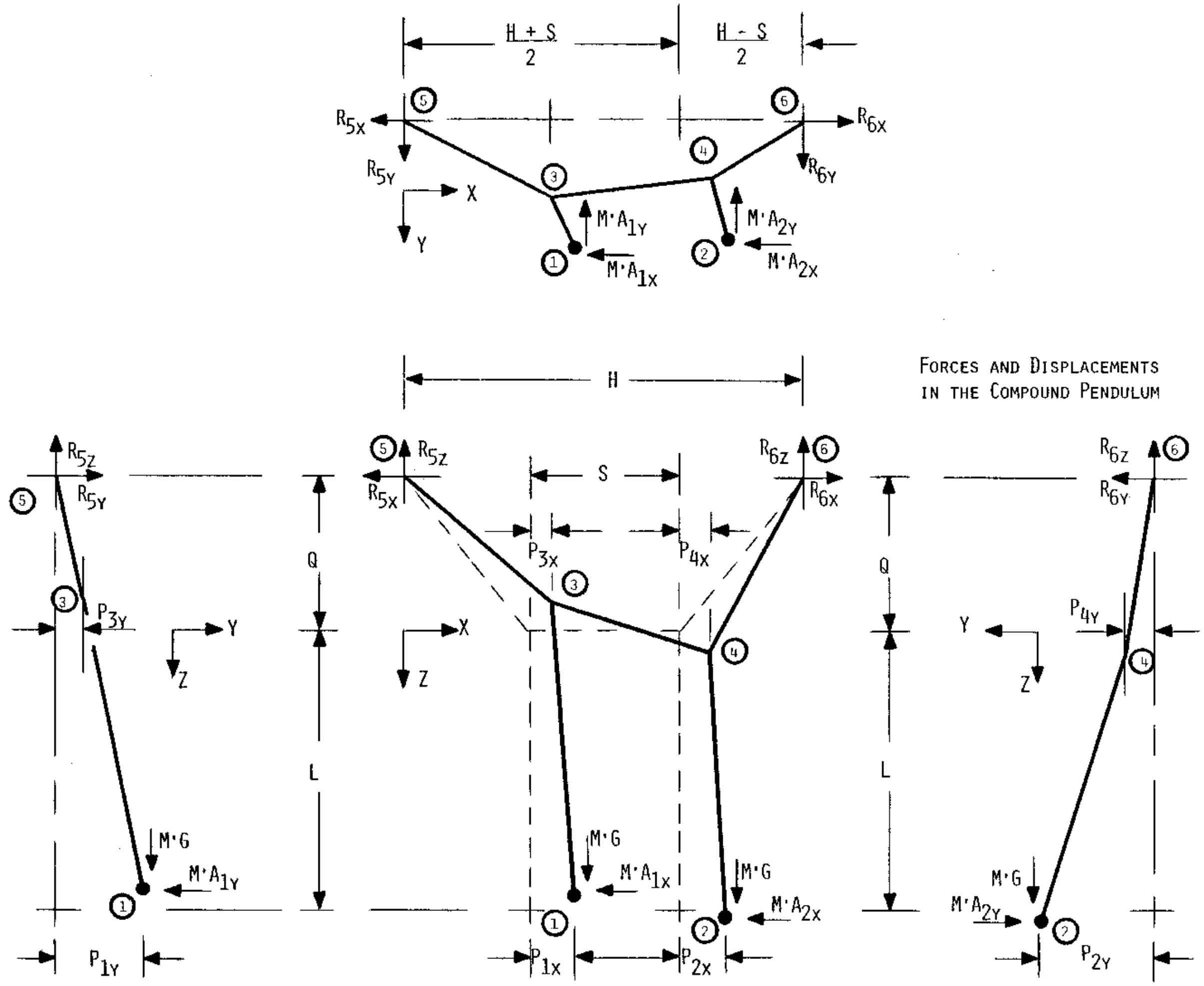
$$\text{Eqa. 6-11:} \quad \sum F_y \xrightarrow{+} = 0 = R_{5y} + R_{6y} + M\cdot(A_{1y} + A_{2y})$$

A similar equation is required to fix the supports along the X-axis:

$$\text{Eqa. 6-12:} \quad \sum F_x \xrightarrow{+} = 0 = R_{5x} - R_{6x} + M\cdot(A_{1x} + A_{2x})$$

At this point we have established twelve equations and introduced fourteen variables: Six components of reaction, R_{5x}, R_{5y}, R_{5z}, R_{6x}, R_{6y}, and R_{6z}; the four variables which specify the positions of the intermediate node points, P_{3x}, P_{3y}, P_{4x}, and P_{4y}; and the four variables which specify the positions of the pendulum weights, P_{1x}, P_{1y}, P_{2x}, and P_{2y}. (Recall that all Z-displacements have been disregarded.) These equations have been chosen so that they comprise a system which decomposes to govern the movements along the two axes of motion. These two systems both involve the vertical components of reactions; so the six equations that could be applied to the movements along a single coordinate axis must, in fact, explain eight variables:

Unknowns	X motion	Y motion
1.	R_{5z}	R_{5z}
2.	R_{6z}	R_{6z}
3.	R_{5x}	R_{5y}
4.	R_{6x}	R_{6y}
5.	P_{3x}	P_{3y}
6.	P_{4x}	P_{4y}
7.	P_{1x}	P_{1y}
8.	P_{2x}	P_{2y}



FORCES AND DISPLACEMENTS IN THE COMPOUND PENDULUM

We can begin to make up our shortage of equations by a statement of vertical equilibrium:

Eqa. 6-13: $R_{5z} + R_{6z} = 2 \cdot M \cdot G$

which can be applied to both X- and Y-motion. One additional equation is available from the consideration of external equilibrium, viz, the balance of moments in the X-Y plane:

Eqa. 6-14: $$\sum M_{⑤} + \circlearrowleft = 0 = - H \cdot R_{6y} - M \cdot A_{1y} \cdot ((H - S)/2 + P_{1x}) - M \cdot A_{2y} \cdot ((H + S)/2 + P_{2x}) + M \cdot A_{1x} \cdot P_{1y} + M \cdot A_{2x} \cdot P_{2y}$$

But we must note that of the six such equations that we have presented, any five are sufficient to specify the sixth. The remaining equations that are required to fully specify X- and Y-motion must arise from our earlier consideration of analytical principles. To completely specify movements along the X-axis, we must introduce the requirement that the X-distance between the intermediate nodes must remain constant:

Eqa. 6-15: $P_{3x} = P_{4x}$

We have established that, in our approximate view of the compound pendulum, displacements in the Y-direction do nothing to change the vertical reactions:

Eqa. 6-16: $R_{5z} = R_{6z}$ ($= M \cdot G$)

We have also introduced the suspicion that X-movements will leave this equation unchanged; but, having introduced the vertical reactions as variables in the system of equations which specify X-motion, we are in a position to allow this equation to manifest itself as being valid through all the behavior modes of the compound pendulum.

Article 7: X-motion. The rigorous, general solution of our non-linear system of eight equations in eight unknowns can be successfully appraoched using a number of schemes of substitution for combinations of variables. Recalling some of the least incisive of these, our essay might have been a hundred pages longer had we not discovered that the pendulum weights <u>always move in a simple harmonic manner relative to one another</u> when considered from the standpoint of movements along a single coordinate axis. To demonstrate and take advantage of this regularity, let us introduce the following change of notation:

Eqa. 7-1: $a_1 = A_{1x} + A_{2x}$

Eqa. 7-2: $a_2 = A_{1x} - A_{2x}$

Eqa. 7-3: $p_1 = P_{1x} + P_{2x}$

Eqa. 7-4: $p_2 = P_{1x} - P_{2x}$

Re-stating the pendulum equations, 6-1 and 6-2, in terms of our new variables yields one simple harmonic equation if we also invoke the equality between P_{3x} and P_{4x}:

Eqa. 7-5: $a_1 = -(G/L) \cdot (p_1 - P_{3x} - P_{4x})$

Eqa. 7-6: $a_2 = -(G/L) \cdot p_2$

Equation 6-10 is also simplified by our change in variables:

Eqa. 7-7: $R_{6z} \cdot H = a_1 \cdot M \cdot (L + Q) + M \cdot G \cdot (H + p_1)$

as can Equation 6-12:

Eqa. 7-8: $R_{6x} = R_{5x} + M \cdot a_1$

Our strategy will now be to derive expressions for P_{3x} and P_{4x} by a combining of Equations 6-5 and 6-6 for these variables:

Eqa. 7-9: $$P_{3x} = \frac{Q \cdot R_{5x}}{R_{5z}} - \frac{H - S}{2}$$

Eqa. 7-10: $$P_{4x} = \frac{-Q \cdot R_{6x}}{R_{6z}} + \frac{H - S}{2}$$

Adding Equations 7-9 and 7-10 yields:

Eqa. 7-11: $$P_{3x} + P_{4x} = Q \cdot \left[\frac{R_{5x}}{R_{5z}} - \frac{R_{6x}}{R_{6z}}\right]$$

This equation can be re-stated entirely in terms of the reactions at node (6) through Equations 7-8 and 6-13:

Eqa. 7-12: $$P_{3x} + P_{4x} = Q \cdot \left[\frac{R_{6x} - M \cdot a_1}{2M \cdot G - R_{6z}} - \frac{R_{6x}}{R_{6z}}\right]$$

Another substitution of Equation 7-10 and re-arrangement will eliminate R_{6x}:

Eqa. 7-13: $$P_{3x} + P_{4x} = \frac{(R_{6z} - M \cdot G) \cdot (H - S - 2P_{4x}) - M \cdot Q \cdot a_1}{2M \cdot G - R_{6z}}$$

Recalling from Equation 6-15 that P_{3x} can be substituted for P_{4x}, allows us to combine Equations 7-13 and 7-5:

Eqa. 7-14: $(L/G) \cdot a_1 + p_1 = (H - S) \cdot (R_{6z} - M \cdot G)/(M \cdot G) - Q \cdot a_1/G$

This equation is simple enough to allow a useful substitution of Equation 7-7 for the purpose of eliminating the last component of reaction, R_{6z}:

Eqa. 7-15: $a_1 \cdot (L + Q)/G + p_1 = (1 - S/H) \cdot (a_1 \cdot (L + Q)/G + p_1)$

Two solutions to this equation which we may immediately note and disregard are S = H and S = 0. Rearranging Equation 7-15 with the intent of preserving the time-variant behaviors we arrive at another equation for simple harmonic (relative) motion:

Eqa. 7-16: $a_1 = -G \cdot p_1 / (L + Q)$

While this equation is at hand, let us use it to substitute for the a_1 of Equation 7-7 in order to establish the behavior of the vertical reactions during displacements along the X-axis:

Eqa. 7-17: $$R_{6z} \cdot H = \frac{-G \cdot p_1}{(L + Q)} \cdot M\ (L + Q) + M \cdot G \cdot (H + p_1) = M \cdot G \cdot H$$

Here we have a validation of our postulate that each vertical reaction retains a constant value of M·G even while the pendulum weights are displaced along the X-axis.

Now let us return to a consideration of Equations 7-6 and 7-16, which simultaneously specify the positions of the pendulum weights along the X-axis. Continuing with our practice of disregarding motions which are initiated by any means except in initial displacement of the pendulum weights, we can solve these two equations by inspection:

Eqa. 7-18: $p_1 = P_{1x} + P_{2x} = C_{1x} \cdot \cos(w_{2x} \cdot t)$

Eqa. 7-19: $p_2 = P_{1x} - P_{2x} = C_{2x} \cdot \cos(w_{1x} \cdot t)$

where C_{1x} and C_{2x} are the initial relative displacements of the two pendulum weights and w_{1x} and w_{2x} have the same identity as they were given in Article 4:

Eqa. 7-20: $$w_{1x} = \sqrt{\frac{G}{L}}$$

Eqa. 7-21: $$w_{2x} = \sqrt{\frac{G}{L + Q}}$$

Equations 7-18 and 7-19 can be solved together to yield separate equations for the positions of the pendulum weights throughout time:

Eqa. 7-22: $2P_{1x} = C_{1x} \cdot \cos(w_{2x} \cdot t) + C_{2x} \cdot \cos(w_{1x} \cdot t)$

Eqa. 7-23: $2P_{2x} = C_{1x} \cdot \cos(w_{2x} \cdot t) - C_{2x} \cdot \cos(w_{1x} \cdot t)$

In order to link these equations to the analysis which proceeded from Equations 2-1 and 2-2, let us evaluate the above in terms of the initial conditions that gave rise to the observations that were made in Article 2: At time equals zero, P_1 was equal to D and P_2 was zero. Combining these considerations with Equations 7-22 and 7-23 will specify $C_{1x} = C_{2x} = D$:

$$\text{Eqa. 7-24:}\quad P_{1x} = \frac{D}{2} \cdot [\cos(w_{2x} \cdot t) + \cos(w_{1x} \cdot t)]$$

$$\text{Eqa. 7-25:}\quad P_{2x} = \frac{D}{2} \cdot [\cos(w_{2x} \cdot t) - \cos(w_{1x} \cdot t)]$$

Using the trigonometric identities for the sums and differences of angles that were introduced in Article 2, we can transform each of the above into the standard form for bi-harmonic motion:

$$\text{Eqa. 7-26:}\quad P_{1x} = D \cdot \cos\left[\frac{(w_{1x} + w_{2x})}{2} \cdot t\right] \cdot \cos\left[\frac{(w_{1x} - w_{2x})}{2} \cdot t\right]$$

$$\text{Eqa. 7-27:}\quad P_{2x} = D \cdot \sin\left[\frac{(w_{1x} + w_{2x})}{2} \cdot t\right] \cdot \sin\left[\frac{(w_{1x} - w_{2x})}{2} \cdot t\right]$$

where D is the initial displacement of the first pendulum weight, and the second pendulum weight starts from rest.

This article has now validated the speculative discourse of Articles 2 and 4 on the basis of the laws of motion and our principles of analysis. We have also established that our principles of analysis imply that vertical reactions are not altered by displacements in the plane of motion and that our mathematical approximation of a compound pendulum is, therefore, linear and elastic in regard to both X- and Y-motion. Hence, we may now claim that our principles of analysis contain the sanction for our linear superposition of the effects that the motion of one pendulum weight has on the other; and for our decomposition of all motions into X- and Y-components.

Article 8: Y-motion. From our experience in the previous article, we should expect that a change of variables which emphasizes the displacements of the pendulum weights relative to one another will greatly simplify our analysis of motion along the Y-axis.

$$\text{Eqa. 8-1:}\quad a_1 = A_1y + A_{2y}$$

$$\text{Eqa. 8-2:}\quad a_2 = A_{1y} - A_{2y}$$

$$\text{Eqa. 8-3:}\quad p_1 = P_{1y} + P_{2y}$$

$$\text{Eqa. 8-4:}\quad p_2 = P_{1y} - P_{2y}$$

A re-statement of the pendulum equations for Y-motion, 6-3 and 6-4, in terms of our new variables will yield:

$$\text{Eqa. 8-5:}\quad a_1 = -(G/L) \cdot (p_1 - P_{3y} - P_{4y})$$

Eqa. 8-6: $a_2 = -(G/L) \cdot (p_2 - P_{3y} + P_{4y})$

A simple harmonic relationship will appear immediately upon re-stating Equation 6-9 in terms of these new variables:

Eqa. 8-9: $a_1 = -(G/(L + Q)) \cdot p_1$

Equation 6-11 is also greatly simplified by this procedure:

Eqa. 8-10: $a_1 = -(R_{5y} + R_{6y})/M$

Using Equations 6-7 and 6-8, we arrive at the following expressions of the relationsips between the reactions and the intermediate node points:

Eqa. 8-11: $P_{3y} = Q \cdot R_{5y}/R_{5z}$

Eqa. 8-12: $P_{4y} = Q \cdot R_{6y}/R_{6z}$

A full compliment of equations for Y-motion is now to be arrived at as we recall that a constant identity of M·G has been established for each of the vertical reactions.

It is unfortunate that this particular compliment of equations does not contain enough information to specify the variable a_2, while a_1 is completely specified by Equation 8-9 and Equation 8-10 is merely redundant. Clearly, one of these equations must be dropped in favor of a transformation of Equation 6-14:

Eqa. 8-13: $-H \cdot R_{6y} = M \cdot A_{1y} \cdot \frac{(H - S)}{2} + M \cdot A_{2y} \cdot \frac{(H + S)}{2}$

Here we have ignored moments which arise from displacements of the pendulum weights along the X-axis in accordance with the principles of analysis that imply a complete separation between actions which take place along the axes of motion. (Please recall from Article 5 that the practical implications of ignoring X-displacements in the moment equation for the X-Y plane was that moments arising from accelerations in the X-direction result in off-setting moments which arise from changing the length of the moment arms through which the Y-forces are acting.) Re-expressing Equation 8-13 in terms our change in variables yields:

Eqa. 8-14: $R_{6Y} = -(M/2) \cdot (a_1 - S \cdot a_2/H)$

Let us now eliminate the reaction forces from our system by combining Equations 8-10, 8-11, 8-12 and 6-16 into the following:

Eqa. 8-15: $Q \cdot a_1 = -G \cdot (P_3 + P_4)$

This equation allows a substitution for " $- P_3 - P_4$" in Equation 8-5:

Eqa. 8-16: $a_1 = -(G/L) \cdot (p_1 + Q \cdot a_1/G)$

which can be re-arranged in the exact form of Equation 8-9, thereby admitting this equation back into our system for Y-motion while allowing us to drop our consideration of moments in the Y-Z plane which had produced this equation by direct observation.

Having thus reconstituted our system of equations for Y-motion, we can now use Equation 8-14 in our search for a simple equation in the variable a_2. Let us begin by eliminating R_{5y} from Equations 8-10, 8-11 and 8-12:

Eqa. 8-17: $M \cdot G \cdot (P_{4y} - P_{3y})/Q = 2R_{6y} + M \cdot a_1$

Eliminating R_{6y} from this equation through a substitution with Equation 8-14 will also eliminate the variable a_1:

Eqa. 8-18: $P_4 - P_3 = \frac{Q}{G} \cdot \frac{S}{H} \cdot a_2$

which can serve to eliminate the references to intermediate nodes in Equation 8-6:

Eqa. 8-19: $a_2 = -(G/(L + S \cdot Q/H)) \cdot p_2$

With Equations 8-9 and 8-19 in hand, we can appeal to the analysis which closed our previous article on X-motion, and say that the frequencies for the simple harmonic motion specified in these two equations can be identified as the internal frequencies of the bi-harmonic oscillator equations that were set out in Articles 2 and 3:

Eqa. 8-20: $w_{1y} = \sqrt{\frac{G}{L + S \cdot Q/H}}$

Eqa. 8-21: $w_{2y} = \sqrt{\frac{G}{L + Q}}$

This means that we can claim that our speculations in Article 3 regarding Y-motions have been validated by the extension of our principles of analysis.

Article 9: Summary. This classical differential form of the equation for a bi-harmonic oscillator would be a system such as the following:

Eqa. 9-1: $A_1 = -\xi P_1 - \zeta P_2$

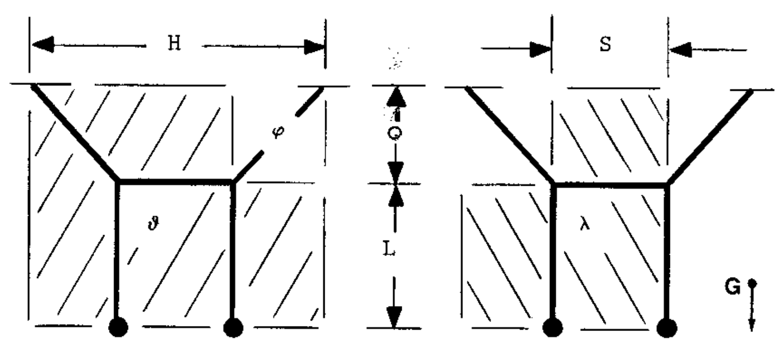
Eqa. 9-2: $A_2 = -\zeta P_1 - \xi P_2$

which would conceal the natural frequencies of whatever system they describe thusly:

Eqa. 9-3: $\xi = \frac{w_1^2 + w_2^2}{2}$

Eqa. 9-4: $\zeta = \frac{w_1^2 - w_2^2}{2}$

The solution that we have presented did not make use of a system such as the above because we opted for a change of variables that would permit a much more compact exposition that would have been possible otherwise. Since this important interpretation of the compound pendulum has been left out of our presentation thus far, let us at least use it as a means to recapitulate our findings. To recall the geometric parameters of the pendulum that seem to be critical to its motion, we would present:



The natural frequencies which we have adduced for the compound pendulum (referring to the figure above) are:

Eqa. 9-5: $w_o = \sqrt{G/(L + Q)}$

Eqa. 9-6: $w_y = \sqrt{G/(L + Q \cdot S/H)}$

Eqa. 9-7: $w_x = \sqrt{G/L}$

We have changed our notation slightly to take advantage of our observation of a single frequency, $w_o = w_{2x} = w_{2y}$, which underlies movements in both coordinate directions. Let us also re-introduce our interpretation of the natural frequencies which determine Y-motion in terms of the areas that are marked out by the nodes of the pendulum:

Eqa. 9-8: $w_y = \sqrt{\frac{G \cdot (\vartheta + \varphi)}{L\vartheta + Q\lambda}}$

Eqa. 9-9: $w_o = \sqrt{\frac{G \cdot (\vartheta - \varphi)}{L\vartheta + Q\lambda}}$

Using this notation, we can set out the parameters of the differential system which introduced this article as follows:

Eqa. 9-10: $\xi_y = \frac{G\vartheta}{L\vartheta + Q\lambda}$

Eqa. 9-11: $\zeta_y = \frac{G\varphi}{L\vartheta + Q\lambda}$

A similar statement for X-motion will not be as elegant because, as we observed earlier, the horizontal dimensions of the compound pendulum do not enter into the frequencies of motion along this axis:

Eqa. 9-12: $\xi_x = \frac{G}{2} \cdot \left[\frac{(L + Q) + L}{L \cdot (L + Q)}\right]$

Eqa. 9-13: $\zeta_x = \frac{G}{2} \cdot \left[\frac{(L + Q) - L}{L \cdot (L + Q)}\right]$

Now we have identified the natural frequencies of the compound pendulum in terms of the parameters of a differential system, Equations 9-1 and 9-2, that can be solved using available techniques, and for any desired boundry conditions. If we restrict ourselves to those motions which arise solely from initial displacements of the pendulum weights, we can write the following equations for the extreme node points:

Eqa: 9-14:
$$\underline{P}_1(t) = \hat{i} \cdot D_{1x} \cdot \cos\left[\frac{w_x + w_o}{2} \cdot t\right] \cdot \cos\left[\frac{w_x - w_o}{2} \cdot t\right] + \hat{j} \cdot D_{1y} \cdot \cos\left[\frac{w_y + w_o}{2} \cdot t\right] \cdot \cos\left[\frac{w_y - w_o}{2} \cdot t\right] + \hat{i} \cdot D_{2x} \cdot \sin\left[\frac{w_x + w_o}{2} \cdot t\right] \cdot \sin\left[\frac{w_x - w_o}{2} \cdot t\right] + \hat{j} \cdot D_{2y} \cdot \sin\left[\frac{w_y + w_o}{2} \cdot t\right] \cdot \sin\left[\frac{w_y - w_o}{2} \cdot t\right]$$

Eqa. 9-15:
$$\underline{P}_2(t) = \hat{i} \cdot D_{2x} \cdot \cos\left[\frac{w_x + w_o}{2} \cdot t\right] \cdot \cos\left[\frac{w_x - w_o}{2} \cdot t\right] + \hat{j} \cdot D_{2y} \cdot \cos\left[\frac{w_y + w_o}{2} \cdot t\right] \cdot \cos\left[\frac{w_y - w_o}{2} \cdot t\right] + \hat{i} \cdot D_{1x} \cdot \sin\left[\frac{w_x + w_o}{2} \cdot t\right] \cdot \sin\left[\frac{w_x - w_o}{2} \cdot t\right] + \hat{j} \cdot D_{1y} \cdot \sin\left[\frac{w_y + w_o}{2} \cdot t\right] \cdot \sin\left[\frac{w_y - w_o}{2} \cdot t\right]$$

In these equations, D_{1x}, D_{2x}, D_{1y}, and D_{2y} are the initial displacements of the respective nodes along their respective axes, while $\hat{i}$ and $\hat{j}$ are vectors of unit length along the X- and Y-axes.